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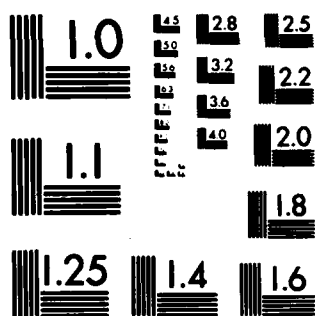
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**CORRECTED DIFFUSION APPROXIMATIONS  
AND THEIR APPLICATIONS**

by

**David Siegmund  
Stanford University**

**TECHNICAL REPORT NO. 24  
DECEMBER 1983**

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## CORRECTED DIFFUSION APPROXIMATIONS AND THEIR APPLICATIONS

David Siegmund  
Stanford University

### 1. Introduction and Summary

Let  $x_1, x_2, \dots$  be independent, identically distributed random variables with mean  $\mu$ , and put  $S_n = x_1 + \dots + x_n$ . In a variety of contexts, e.g. sequential analysis, nonparametric statistics, queueing theory, insurance risk theory, one is interested in a probability or expected value defined in terms of the time at which  $S_n$  first crosses a boundary  $c(n)$ . In general  $c(n)$  is nonlinear, but the linear case is particularly important. Exact results are possible only in very special circumstances, but one can obtain asymptotic approximations reasonably generally.

One possible formulation involves contraction of the time and space coordinates by the factor  $m^{-1}$  and consideration of the asymptotic behavior of (for example)

$$(1) \quad P\{S_n/m \geq c(n/m) \text{ for some } 1 \leq n \leq m\}$$

as  $m \rightarrow \infty$ . If  $c(0) > 0$  and  $c$  is concave, it is clear from the law of large numbers that (1) converges to 0 if  $\mu < c(1)$  and to 1 if  $\mu > c(1)$ . Precise asymptotic approximations to (1) have been developed largely in the context of sequential analysis. See [23] or [27] for a summary of results and statistical applications. Since these approximations involve the rate at which a sequence of probabilities tends to 0, it seems appropriate to call them large deviation approximations.

A second possibility is to contract time by the factor  $m^{-1}$  and space by  $m^{-1/2}$ , so that instead of (1) one considers

$$(2) \quad P\{S_n/m^{1/2} \geq c(n/m) \text{ for some } 1 \leq n \leq m\}$$

with the additional assumption that  $\mu = \mu_m \sim \xi/m^{1/2}$  for some fixed  $-\infty < \xi < \infty$ . Then under quite general conditions (2) converges as  $m \rightarrow \infty$  to

$$(3) \quad P\{W(t) \geq c(t) \text{ for some } 0 \leq t \leq 1\},$$

where  $W(t)$  is Brownian motion with drift  $\xi$ . Unfortunately the approximation of (2) by (3) is rather crude unless  $m$  is quite large, so it seems interesting to try to develop an asymptotic expansion of (2) with (3) as the leading term. Such approximations resemble Edgeworth expansions to improve the central limit theorem. They have been called corrected diffusion approximations [21].

The purpose of this paper is to survey the present state of development of corrected diffusion approximations and to compare them with the considerably more developed large deviation approximations.

Section 2 reviews the ruin problem of risk theory, for which the classical large deviation approximation is due to Cramér (cf. Feller, [7], p. 411). Section 3 discusses some related results for which a large deviation formulation is in a certain sense inappropriate. Section 4 is concerned with first passage times to a linear boundary with applications to the one sample Kolmogorov-Smirnov statistic, truncated sequential probability ratio tests, and risk theory.

The results of Sections 2-4 involve crossings of linear boundaries, for which corrected diffusion approximations are typically as good and

often better than large deviation approximations. However, it has proved difficult to develop corrected diffusion approximations for nonlinear boundaries, and it is only recent research of Hogan [9] that has led to some progress. Section 5 discusses one particular application of Hogan's work to repeated significance tests for a normal mean.

## 2. Ruin of a Risk Process

For  $b > 0$  define

$$\tau = \tau(b) = \inf\{n: S_n \geq b\} ,$$

with the understanding that  $\inf \phi = +\infty$ . In this section we shall review some known results concerning  $P\{\tau(b) < \infty\}$ . With the proper interpretation this is the probability of ultimate ruin of a risk process or the stationary waiting time distribution of a single server queue. See Feller [7], Chapters VI and XII.

Assume that the distribution of the  $x_i$  is of the form

$$(4) \quad dF_\mu(x) = \exp\{\theta x - \psi(\theta)\} dF(x) ,$$

where  $\mu = \psi'(\theta) = \int x dF_\mu(x)$  is strictly increasing (unless  $F$  is concentrated at a point), so one can consider  $\mu$  to be a function of  $\theta$ , or  $\theta$  a function of  $\mu$  - say  $\theta(\mu)$ . After a linear transformation one can assume without loss of generality that

$$(5) \quad \int x dF(x) = 0 , \quad \int x^2 dF(x) = 1 .$$

In particular  $F_0 = F$ . Since  $\psi$  is strictly convex, given any  $\mu_0 < 0$  ( $\mu_1 > 0$ ) there exists at most one value  $\mu_1 > 0$  ( $\mu_0 < 0$ ) such that

$\psi(\theta_0) = \psi(\theta_1)$ , where  $\theta_i = \theta(\mu_i)$ . We shall assume that such a "conjugate" value of  $\mu$  exists for all  $\mu \neq 0$ .

By Wald's likelihood ratio identity, for any stopping time  $N$  and event  $A \in \mathfrak{B}(x_1, \dots, x_N)$ , for all  $\mu \neq \mu'$

$$(6) \quad P_\mu(A \cap \{N < \infty\}) = E_{\mu'}[\exp\{(\theta - \theta')S_N - N[\psi(\theta) - \psi(\theta')]\}; A \cap \{N < \infty\}],$$

where  $\theta = \theta(\mu)$ ,  $\theta' = \theta(\mu')$ , and  $E(Z; B) = E(ZI_B)$ . In particular, if  $\mu_0 < 0 < \mu_1$  are conjugate in the sense of the preceding paragraph, then (6) yields

$$(7) \quad P_{\mu_0}\{\tau < \infty\} = E_{\mu_1}[\exp\{-\Delta(S_\tau - b)\}] \exp(-\Delta b),$$

where  $\Delta = \theta_1 - \theta_0$ . From (7), the renewal theorem, and some calculation, it follows that as  $b \rightarrow \infty$

$$(8) \quad P_{\mu_0}\{\tau < \infty\} \sim K(\mu_0) \exp(-\Delta b),$$

where, for nonarithmetic random variables,

$$(9) \quad K(\mu_0) = P_{\mu_0}\{\tau_+ = \infty\} P_{\mu_1}\{\tau_- = \infty\} / \Delta \mu_1$$

with  $\tau_+$  ( $\tau_-$ ) =  $\inf\{n: S_n > 0$  ( $S_n \leq 0$ )}. In general  $K(\mu_0)$  must be evaluated numerically [27], p. 25. The relation (8) is Cramér's classical approximation obtained essentially by the method introduced by Feller [7], Chapter XII.

Suppose now that  $b \rightarrow \infty$ ,  $\mu_0 \rightarrow 0$  in such a way that  $b\mu_0$  is bounded away from  $-\infty$  and 0. It is easy to see that  $\Delta b$  is bounded away from 0 and  $\infty$ . The following result appears in [21].



(10) Theorem. Suppose  $b \rightarrow \infty$ ,  $\Delta \rightarrow 0$  so that  $\Delta b$  is a fixed positive number. If  $F$  is nonarithmetic, then

$$P_{\mu_0} \{ \tau < \infty \} = \exp(-\Delta b) [1 - \rho_+ \Delta + o(\Delta)] ,$$

where  $\rho_+ = \frac{1}{2} E_0(S_{\tau_+}^2) / E_0(S_{\tau_+})$ . If  $F$  is strongly nonarithmetic in the sense that

$$\liminf_{|\lambda| \rightarrow \infty} |1 - \int e^{i\lambda x} dF(x)| > 0 ,$$

then

$$(11) \quad P_{\mu_0} \{ \tau < \infty \} = \exp(-\Delta b) [1 - \rho_+ \Delta + \frac{1}{2} \rho_+^2 \Delta^2 + o(\Delta^2)] .$$

Remarks.

(12) Since  $1-x + \frac{1}{2}x^2 = \exp(-x) + o(x^2)$  as  $x \rightarrow 0$ , and  $\exp(-x)$  is always positive, it seems reasonable for numerical purposes to express (11) in the form

$$P_{\mu_0} \{ \tau < \infty \} = \exp[-\Delta(b + \rho_+)] + o(\Delta^2) .$$

This also has an appealing interpretation, since  $\exp[-\Delta(b + \rho_+)]$  is exactly the probability that Brownian motion with drift  $-\frac{1}{2}\Delta$  ever crosses the level  $b + \rho_+$ . Most of the results of Sections 3-5 admit similar interpretation.

(13) At first glance (11) is quite surprising, because an informal expansion of (7) leads one to expect the coefficient of  $\Delta^2$  in (11) to involve  $\lim_{b \rightarrow \infty} E_{\mu_1} [(S_{\tau} - b)^2] = E_0(S_{\tau_+}^3) / 3 E_0(S_{\tau_+})$ .

(14) The proof of (11) follows from (19) - (21) given below. For details, see [21].

(15) There is a version of (11) in the case that  $F$  is an arithmetic distribution, provided  $b \rightarrow \infty$  through the span of the support of  $F$ .

(16) In view of (8) and (11), it is not surprising to obtain

$$K(\mu_0) = \exp(-\Delta\rho_+) + o(\Delta^2) \quad (\Delta \rightarrow 0),$$

which follows from (20) and some easy calculation.

(17) An algorithm for computing  $\rho_+$  numerically is given in [21]. For the special case  $dF(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2} x^2)$ ,

$$\rho = -\pi^{-1} \int_0^\infty \lambda^{-2} \log\{2[1 - \exp(-\frac{1}{2} \lambda^2)]/\lambda^2\} d\lambda \approx .583.$$

(18) Because of (16) one expects (and is supported by experience) to find that (8) and (12) give similar numerical results. The main advantage of (12) is that it requires numerical evaluation of only a single constant. The results of Section 3 provide examples where the asymptotic normalization of (8) is difficult to justify, but precise analogues of (12) are easily obtained.

Let  $W(t)$ ,  $0 \leq t < \infty$ , denote Brownian motion with drift  $\mu$  and put

$$\tau_W = \tau_W(b) = \inf\{t: W(t) \geq b\}.$$

Let  $G(t; \mu, b) = P_\mu\{\tau_W(b) \leq t\}$ . Suppose to be definite that  $F$  is non-arithmetic and let

$$H(x) = (E_0 S_{\tau_+})^{-1} \int_{(0, x)} P_0\{S_{\tau_+} \geq y\} dy.$$

The following lemmas yield (10) as an immediate consequence.

(19) Lemma. Suppose  $F$  is nonarithmetic. Let  $\mu \rightarrow 0$ ,  $b \rightarrow \infty$  in such a way that for some  $-\infty < \xi < \infty$ ,  $\mu b \rightarrow \xi$ . Then for all  $0 \leq t$ ,  $x \leq \infty$

$$\lim_{\mu} P_{\mu} \{ \tau(b) \leq b^2 t, S_{\tau} - b \leq x \} = G(t; \xi, 1) H(x),$$

and for all  $r > 0$

$$\lim_{\mu} E_{\mu} \{ (S_{\tau} - b)^r; \tau < \infty \} = E_0(S_{\tau_+}^{r+1}) / (r+1) E_0(S_{\tau_+}).$$

(20) Lemma. Assume that  $F$  is nonarithmetic. Let  $r \geq 0$ ,  $\mu > 0$ . Then

$$\lim_{\mu \rightarrow 0} \mu E_{\mu} \{ \tau_+, S_{\tau_+}^r \} = E_0(S_{\tau_+}^{r+1}) / (r+1)$$

and consequently as  $\mu \rightarrow 0$

$$E_{\mu}(S_{\tau_+}^r) = E_0(S_{\tau_+}^r) + \theta r E_0(S_{\tau_+}^{r+1}) / (r+1) + o(\mu),$$

where  $\theta = \theta(\mu)$ .

(21) Lemma. Suppose  $F$  is strongly nonarithmetic. There exists an  $\epsilon > 0$  such that uniformly for  $\mu \in [0, \epsilon]$

$$E_{\mu}[(S_{\tau} - b)^j] = E_{\mu}(S_{\tau_+}^{j+1}) / (j+1) E_{\mu}(S_{\tau_+}) + O(e^{-\epsilon b})$$

for  $j=1, 2, 3$ .

### 3. Two-Sided Problems

Consider the stopping rule of a sequential probability ratio test

$$(22) \quad T = \inf\{n: S_n \notin [a, b]\},$$

where  $a \leq 0 < b$ . If  $b \rightarrow \infty$  and  $a \rightarrow -\infty$ , then for fixed  $\mu_0 < 0$ ,  $P_{\mu_0}\{S_T \geq b\} \sim P_{\mu_0}\{\tau(b) < \infty\}$ , so a large deviation approximation does not distinguish between this situation and that of Section 2. In [18] I have a heuristic approximation to  $P_{\mu_0}\{S_T \geq b\}$  which did not lose sight of the lower boundary at  $a$ , but which seems impossible to justify in general. The following approximations do have precise mathematical justification and agree with the heuristic approximations of [18] when  $\mu \rightarrow 0$ .

(23) Theorem. Assume that  $F$  is nonarithmetic. Suppose  $b \rightarrow \infty$ ,  $a \rightarrow -\infty$ , and  $\Delta = \theta_1 - \theta_0 \rightarrow 0$  in such a way that  $\Delta b$  and  $b/|a|$  are fixed positive numbers. Then for  $j=0$  or  $1$

$$(24) \quad P_{\mu_j}\{S_T \geq b\} = \frac{1 - \exp[(-1)^j \Delta(a + \rho_-)]}{\exp[(-1)^j \Delta(b + \rho_+)] - \exp[(-1)^j \Delta(a + \rho_-)]} + o(\Delta),$$

where  $\rho_{\pm} = \frac{1}{2} E_0(S_{\tau_{\pm}}^2) / E_0(S_{\tau_{\pm}})$  and  $\tau_{+} (\tau_{-}) = \inf\{n: S_n > 0 (S_n \leq 0)\}$ .

Also

$$E_{\mu_j}(T) = \mu_j^{-1}[(a + \rho_-) + (b + \rho_+ - a - \rho_-) p^*] + o(\Delta^{-1}),$$

where  $p^*$  denotes the right hand side of (24). If  $F$  is strongly nonarithmetic, the remainder in (24) is  $o(\Delta^2)$ .

#### Remarks.

(25) Like (12), (23) has an interesting interpretation in terms of Brownian motion: The approximations would be exact for Brownian motion with drift  $\pm \frac{1}{2}\Delta$  stopped upon hitting the boundaries  $b + \rho_+$  or  $a + \rho_-$ .

(26) For the case  $\mu=0$ , see [18].

(27) Example.

Suppose  $y_1, y_2, \dots$  are independent with probability density function  $\lambda e^{-\lambda x}$  for  $x \geq 0$ . A sequential probability ratio test of  $H_0: \lambda = \lambda^{(0)}$  against  $H_1: \lambda = \lambda^{(1)}$  ( $\lambda^{(0)} < \lambda^{(1)}$ ) is defined by a stopping rule of the form (22) with  $S_n = \sum_{i=1}^n (1 - \lambda^* y_i)$ , where  $\lambda^* = (\lambda^{(1)} - \lambda^{(0)}) / \log(\lambda^{(1)} / \lambda^{(0)})$ . For this test the classical approximations of Wald [26] for the error probabilities and expected sample size are given by (23), but with  $\rho_+ = \rho_- = 0$ . From the lack of memory property of the exponential distribution it follows that  $S_{\tau_-}$  is exponentially distributed and hence that  $\rho_- = -1$ . An easy calculation based on the Wiener-Hopf factorization ([7], p. 605) shows that  $\rho_+ = 1/3$ . Hence (23) provides a theoretical explanation for the modifications to Wald's approximations suggested independently in [8] and [12]. (The modification of  $a$  to  $a-1$  is also easily justified by relating this test to an equivalent one for the mean of a Poisson process observed continuously, for which there is no excess at the lower boundary. But the suggestion to replace  $b$  by  $b + 1/3$  has apparently been justified on the basis of numerical evidence.)

Now suppose that  $T$  is defined by (22) with  $a=0$ . This stopping rule arises in the study of cusum control charts [5], where one quantity of particular interest is  $E_{\mu_j}(T) / P_{\mu_j}\{S_T \geq b\}$ , the so-called average run length.

(28) Theorem. Suppose  $a=0$  and  $F$  is nonarithmetic. Let  $b \rightarrow \infty$  and  $\Delta \rightarrow 0$  so that  $\Delta b$  is a fixed positive number. Then for  $j=0$  or  $1$

$$\begin{aligned} E_{\mu_j}(T) / P_{\mu_j}\{S_T \geq b\} &= (\Delta |\mu_j|)^{-1} \{ \exp[(-1)^j \Delta(b + \rho_+ - \rho_-)] \\ &\quad - (-1)^j \Delta(b + \rho_+ - \rho_-) - 1 \} + o(\Delta^{-1}), \end{aligned}$$

where  $\rho_+$  and  $\rho_-$  are defined as in (23).

Remarks.

(29) To see that this approximation provides excellent numerical results it can be compared with the exact numerical computations of van Dobben de Bruyn [5] for the case of the normal distribution and Lorden and Eisenberger [15] for the exponential distribution.

(30) The proof of (28) uses (20) repeatedly; see [23] for details.

(31) An interpretation of (28) in terms of Brownian motion involves the replacement of  $b$  by  $b + \rho_+ - \rho_-$ . Without this correction the approximation is very poor for  $\mu < 0$ .

Assume now that  $dF(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2} x^2)$ , so the  $P_\mu$  distribution of  $x_1, x_2, \dots$  is normal with mean  $\mu$  and variance 1. A special but particularly interesting case of a class of sequential tests proposed by Anderson [ ] is based on the stopping rule

$$\tilde{T} = \inf\{n: |S_n| \geq b - \frac{1}{2} n\} \quad (\eta > 0) .$$

Anderson studied these tests for Brownian motion. In general a corrected diffusion approximation for the error probabilities is quite complicated, but for  $\mu = -\eta$  (with a similar result for  $\mu = \eta$ ), if  $b \rightarrow \infty, \eta \rightarrow 0$  so that  $b\eta$  is fixed, then

$$(32) \quad P_{-\eta}\{\tilde{S}_{\tilde{T}} > 0\} = \frac{1}{2} \exp[-\eta(b + \rho_+)] + o(\eta) ,$$

where  $\rho_+ \approx .583$  (cf. (17)). It is interesting to compare (32) with the approximation of Lorden [14], who on the basis of numerical computations and curve fitting suggested

$$\exp(-\eta b) [.4996 - .28645 \eta + .0696 \eta^2] ,$$

which in the range he explored is extremely accurate. For  $\rho_+ = .583$ , the right hand side of (32) equals

$$\exp(-\eta b) [.5 - .2915 \eta + .0850 \eta^2 + o(\eta^2)] ,$$

and numerically (32) is usually even closer to Lorden's approximation than this expansion suggests.

#### 4. The Distribution of $\tau(b)$ .

Assume that  $x_1, x_2, \dots$  are independent, identically distributed random variables with mean 0, variance 1, and finite third moment  $\kappa$ . The exponential family assumption (4) will be introduced when needed. Let  $S_n = x_1 + \dots + x_n$  and

$$\tau = \inf\{n: S_n \geq b\} .$$

Let  $\hat{f}(\lambda) = E[\exp(i\lambda x_1)]$ , and assume that for some  $n_0 \geq 1$

$$(33) \quad \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^{n_0} d\lambda < \infty .$$

It follows from (33) that for all  $n \geq n_0$   $S_n$  has a bounded density function which obeys a variety of local limit theorems (cf. (45) below). It will be convenient to use the notation

$$P_{\xi}^{(m)}(A) = P(A | S_m = \xi) , \quad A \in \mathcal{B}(x_1, \dots, x_m) .$$

(34) Theorem. Suppose  $b = m^{\frac{1}{2}}\zeta$  and  $\xi = m^{\frac{1}{2}}\xi_0$  for some  $\zeta > 0$  and  $-\infty < \xi_0 < \zeta$ . Then as  $m \rightarrow \infty$

$$(35) \quad P_{\xi}^{(m)}\{\tau < m\} = \exp\{-2(b+\rho_+)(b+\rho_+ - \xi - \kappa/3)/(m + \kappa\xi/3)\} + o(m^{-1/2}) .$$

If in addition  $c = m^{1/2}\gamma$  for some  $\gamma \leq \zeta$ , then

$$(36) \quad P\{\tau < m, S_m < c\} = \Phi\left(\frac{c + \kappa/3 - 2(b+\rho_+)}{(m + \kappa c/3)^{1/2}}\right) + o(m^{-1/2}) .$$

Here  $\kappa = E(x_1^3)$ ,  $\rho_+ = \frac{1}{2} E(S_{\tau_+}^2)/E(S_{\tau_+})$ , and  $\Phi$  denotes the standard normal distribution function.

(37) Theorem. Now assume that the distribution of  $x_1$  has the exponential form (4) and that for some  $n_0 \geq 1$  and all  $\mu$  in a neighborhood of 0

$$\int_{-\infty}^{\infty} |E_{\mu}[\exp(i\lambda x_1)]|^{n_0} d\lambda < \infty .$$

Let  $\mu_0 < 0 < \mu_1$  be such that  $\psi(\theta_1) = \psi(\theta_0)$ , where  $\theta_i = \theta(\mu_i)$ , and put  $\Delta = \theta_1 - \theta_0$ . Suppose that  $b = m^{1/2}\zeta$ ,  $c = m^{1/2}\gamma$  for some  $\zeta > 0$ ,  $\gamma \leq \zeta$ , and that  $m^{1/2}\Delta = \delta$  is a fixed positive constant. Then as  $m \rightarrow \infty$ , for  $j=0$  or 1

$$P_{\mu_j}\{\tau < m, S_m < c\} = \exp[-(-1)^j \Delta(b+\rho_+)] \Phi\left(\frac{c+\kappa/3 - 2(b+\rho_+)}{(m+\kappa c/3)^{1/2}} + \frac{1}{2}(-1)^j \Delta(m+\kappa c/3)^{1/2}\right) + o(m^{-1/2}) .$$

#### Remarks.

(38) Guided by the results of Sections 2 and 3, the approximations (35)-(37) have been expressed insofar as possible to resemble analogous exact results for Brownian motion. In fact, they are precisely the corresponding results for Brownian motion with drift 0 or  $\pm \frac{1}{2}\Delta$ ,  $b$  replaced by  $b + \rho_+$ ,  $\xi(c)$  replaced by  $\xi + \kappa/3$  ( $c + \kappa/3$ ), and  $m$  replaced by  $m + \xi\kappa/3$  ( $m + \kappa c/3$ ).



(39) The approximations (36), (37) are similar but not identical to those suggested in [21] on the basis of some heuristic Laplace transform inversions. To illustrate the difference consider

$$(40) \quad P_{\mu}\{\tau \leq m\} = P_{\mu}\{S_m \geq b\} + P_{\mu}\{\tau < m, S_m < b\}.$$

Suppose for simplicity that  $F=\Phi$  is standard normal, so  $F_{\mu}(x) = \Phi(x-\mu)$ . Then the first term on the right hand side of (40) is exactly  $1-\Phi(bm^{-1/2} - \mu m^{1/2})$ , and the second can be approximated by (37) to obtain

$$P_{\mu}\{\tau \leq m\} \approx 1-\Phi(bm^{-1/2} - \mu m^{1/2}) + \exp[2\mu(b+\rho_+)] \Phi[-(b+2\rho_+)m^{-1/2} - \mu m^{1/2}],$$

where  $\rho_+ \approx .583$ . The approximation suggested in [21] is

$$P_{\mu}\{\tau \leq m\} = 1-\Phi[(b+\rho_+)m^{-1/2} - \mu m^{1/2}] + \exp[2\mu(b+\rho_+)] \Phi[-(b+\rho_+)m^{-1/2} - \mu m^{1/2}].$$

The differences between these approximations is  $o(m^{-1/2})$ , so both are consistent with (37). However, in view of (40), the first one would seem preferable, and some numerical experimentation shows it is typically more accurate - especially for small values of  $m$ .

(41) Asymptotic expansions essentially equivalent to (36) and (37) were obtained in [24]; but the method of proof used there broke down in the most interesting special case  $c=b$  ( $\gamma=\zeta$ ).

#### Examples.

(42) Let  $F_n(x)$  denote the empirical distribution function of  $n$  independent, uniform random variables. The representation of uniform order statistics in terms of exponentially distributed random variables (e.g. [3], p. 285) shows that

$$P\left\{\sup_{0 < x < 1} (x - F_n(x)) > n^{-\frac{1}{2}}\zeta\right\} = P\left\{\max_{1 \leq j \leq n} (W_j - j) \geq n^{\frac{1}{2}}\zeta - 1 \mid W_{n+1} - (n+1) = -1\right\},$$

where  $W_j = y_1 + \dots + y_j$  with  $y_1, y_2, \dots, y_{n+1}$  independent standard exponential random variables. Putting  $m = n+1$ ,  $\xi = -1$ ,  $b = n^{\frac{1}{2}}\zeta - 1$ , and  $\rho_+ = 1$  (because  $S_{\tau_+}$  is standard exponential) in (35) yields

$$P\left\{\sup_{0 < x < 1} n^{\frac{1}{2}}(x - F_n(x)) > \zeta\right\} = \exp[-2\zeta(\zeta + (3n^{\frac{1}{2}})^{-1})] + o(n^{-\frac{1}{2}}),$$

a result first derived by Smirnov [25]. This approximation is much simpler and almost as accurate as the large deviation approximation given in [22]. See [22] or [29] for numerical examples.

(43) The approximations (34), (35), and similar results for two-sided stopping rules can be used to calculate error probabilities for truncated sequential probability ratio tests. See [22] and [23] for numerical examples for normal and exponential distributions.

(44) Asmussen [2] has studied various approximations to the probability of ruin before time  $t$  for a risk process, and has concluded that an approximation similar to (37) is better than several alternative proposals.

#### Proof of (34).

Let  $f_n$  denote the probability density function of  $S_n$ , which by (33) exists for all  $n \geq n_0$ . According to [16], p. 207.

$$(45) \quad f_n(xn^{\frac{1}{2}})n^{\frac{1}{2}} = \phi(x)\{1 + (\kappa/6n^{\frac{1}{2}})(x^3 - 3x)\} + (1 + |x|^3)^{-1} o(n^{-\frac{1}{2}}),$$

where  $o(\cdot)$  is uniform in  $x$ .

Let  $P_{\lambda, \xi}^{(m)}(A) = P(A \mid S_0 = \lambda, S_m = \xi)$ . Set  $\lambda = m^{\frac{1}{2}}\lambda_0$  and  $\xi = m^{\frac{1}{2}}\xi_0$  for some  $\lambda_0, \xi_0 < \zeta$ . Let  $\xi' = 2b - \xi = m^{\frac{1}{2}}(2\zeta - \xi_0)$  denote  $\xi$  reflected

about b. From (45) and the renewal theorem it follows as in Lemmas 1-3 of [24] that for  $m_1 = m[1 - (\log m)^{-2}]$  and some  $\varepsilon_m \rightarrow 0$

$$(46) \quad P_{\lambda, \xi}^{(m)}\{\tau < m\} - P_{\lambda, \xi}^{(m)}\{\tau < m_1, S_{\tau-b} < m^{\frac{1}{2}} \varepsilon_m\} = o(m^{-\frac{1}{2}}).$$

and

$$(47) \quad P_{\lambda, \xi}^{(m)}\{\tau < m\} - P_{\lambda, \xi}^{(m)}\{\tau < m_1, S_{\tau-b} < m^{\frac{1}{2}} \varepsilon_m\} = o(m^{-\frac{1}{2}}).$$

Set  $A_m = \{\tau < m_1, S_{\tau-b} < m^{\frac{1}{2}} \varepsilon_m\}$ , and let  $\ell^{(m)}(n, S_n)$  denote the likelihood ratio of  $x_1, \dots, x_n$  under  $P_{\lambda, \xi}^{(m)}$  relative to  $P_{\lambda, \xi'}^{(m)}$ . For all  $n \leq m - n_0$

$$(48) \quad \ell^{(m)}(n, S_n) = \frac{f_{m-n}(\xi - S_n) f_m(\xi' - \lambda)}{f_{m-n}(\xi' - S_n) f_m(\xi - \lambda)}.$$

By (46) and Wald's likelihood ratio identity

$$(49) \quad P_{\lambda, \xi}^{(m)}\{\tau < m\} = E_{\lambda, \xi}^{(m)}\{\ell^{(m)}(\tau, S_{\tau}); A_m\} + o(m^{-\frac{1}{2}}).$$

Substitution of (48) into (49) and expansion with the aid of (45) gives the first order result

$$P_{\lambda, \xi}^{(m)}\{\tau < m\} \rightarrow \exp[-2(\zeta - \lambda_0)(\zeta - \xi_0)].$$

This motivates the following reformulation of (49) (which is justified by (47) and the fact that  $P_{\lambda, \xi}^{(m)}\{\tau = m\} = o(m^{-\frac{1}{2}})$ ):

$$(50) \quad P_{\lambda, \xi}^{(m)}\{\tau < m\} - \exp[-2(\zeta - \lambda_0)(\zeta - \xi_0)] + o(m^{-\frac{1}{2}}) \\ = E_{\lambda, \xi}^{(m)}\{\ell^{(m)}(\tau, S_{\tau}) - \exp[-2(\zeta - \lambda_0)(\zeta - \xi_0)]; A_m\}.$$

The likelihood ratio of  $x_1, \dots, x_n$  under  $P_{\lambda, \xi}^{(m)}$ , relative to  $P(\cdot | S_0 = \lambda)$  is  $f_{m-n}(\xi' - S_n) / f_m(\xi' - \lambda)$ . Hence by (48) and Wald's likelihood ratio identity once again the right hand side of (50) becomes

$$(51) \quad E \left\{ \frac{f_{m-\tau}(\xi - S_\tau)}{f_m(\xi - \lambda)} - \exp[-2(\zeta - \lambda_0)(\zeta - \xi_0)] \frac{f_{m-\tau}(\xi' - S_\tau)}{f_m(\xi' - \lambda)} ; A_m | S_0 = \lambda \right\}.$$

The rest of the proof of (35) involves use of (45) to expand the integrand in (51) and application of (19) to evaluate the resulting expectations. To avoid some tedious algebra, consider the special case  $f_n(x) = n^{-1/2} \phi(xn^{-1/2})$  ( $\phi$  the standard normal density function). Let  $R_m = S_\tau - m^{1/2}\zeta$ . The integrand in (51) equals

$$[(1 - \tau/m)^{1/2} \phi(\xi_0 - \lambda_0)]^{-1} \left\{ \phi\left(\frac{\zeta - \xi_0 + R_m/m^{1/2}}{(1 - \tau/m)^{1/2}}\right) - \phi\left(\frac{\zeta - \xi_0 - R_m/m^{1/2}}{(1 - \tau/m)^{1/2}}\right) \right\},$$

which can be expanded to give

$$(52) \quad -2(1 - \tau/m)^{-1/2} \exp[\frac{1}{2}(\xi_0 - \lambda_0)^2 - \frac{1}{2}(\zeta - \xi_0)^2 / (1 - \tau/m)] [(\zeta - \xi_0) R_m / m^{1/2} (1 - \tau/m)] \\ + O([1 + R_m^2] / m)$$

uniformly on  $A_m$ . According to (19)  $\tau/m$  and  $R_m$  are asymptotically independent, converge in law, and  $(R_m)$  is uniformly integrable. Also (52) is a bounded, continuous function of  $\tau/m$  on  $A_m$ . Hence (52) can be substituted into (51) and (19) applied to evaluate the result. Putting  $\lambda=0$  and performing the appropriate integrations yields (35) in the case of normally distributed  $x_i$ . The general case is similar, but with more complicated algebra.

Formally (36) and (37) follow by substituting (35) into

$$(53) \quad P\{\tau < m, S_m < c\} = \int_{(-\infty, c)} P_{0, \xi}^{(m)}\{\tau < m\} P\{S_m \in d\xi\}.$$

However, some care is required to justify this calculation - especially in the case  $c=b$  ( $\gamma=\zeta$ ), when  $\xi$  in (53) can be arbitrarily close to  $b$ . It is easy to see that (35) holds uniformly on each compact subinterval of  $(-\infty, \zeta)$ ; but if  $\xi_0 \rightarrow \zeta$  (52) is not necessarily bounded, (46) may fail to hold, and indeed the proof of (35) disintegrates.

To circumvent this difficulty, let

$$\tau^* = \sup\{n: n < m, S_n \geq b\},$$

and observe that

$$P_{0, \xi}^{(m)}\{\tau < m\} = P_{0, \xi}^{(m)}\{\tau^* > 0\} = P_{\xi, 0}^{(m)}\{\tau < m\},$$

so to approximate  $P_{0, \xi}^{(m)}\{\tau < m\}$  for  $\zeta - \epsilon \leq \xi_0 < \zeta$ , it suffices to consider  $P_{\lambda, \xi}^{(m)}\{\tau < m\}$  for  $\xi=0$  and  $\zeta - \epsilon \leq \lambda_0 < \zeta$  (recall that  $\lambda_0 = m^{-\frac{1}{2}}\lambda$ ). It is easy to see from (50) - (52) that uniformly for  $\zeta - \epsilon \leq \lambda_0 \leq \zeta - m^{-\frac{1}{2}}$ ,

$$P_{\lambda, 0}^{(m)}\{\tau < m\} = \exp\{-2\zeta(\zeta - \lambda_0)\} + O(m^{-\frac{1}{2}}),$$

which suffices to justify formal substitution of (35) into (53) for  $\xi$  in a neighborhood of  $b$ .

There remains the problem of large  $\xi$ . If sufficiently many moments of  $x_i$  exist, for example, under the exponential family assumption of (37), it is easy to improve (45) by a tilting argument and show that (35) holds uniformly for  $-\log m \leq \xi_0 \leq \zeta - \epsilon$ . This suffices to complete the proof of (37). However, under the minimal assumption of a finite third moment I have not been able to complete the proof of (36) except

by resorting to the rather different techniques developed in [24], which prove (36) directly (for  $\gamma < \zeta$ ) without going through (35).

Remark. It is interesting to compare the preceding proof of (35) with the superficially similar proof of the analogous large deviations result (Theorem 1 of [22]). An especially important difference is that in the large deviation scaling, after an appropriate use of Wald's likelihood ratio identity the distribution of  $\tau/m$  degenerates to a point mass. This means that if  $\tau$  were defined by a nonlinear boundary, one could expect to obtain some results by linearizing the boundary at the point where the distribution of  $\tau/m$  concentrates. In the case of a diffusion limit the distribution of  $\tau/m$  does not degenerate, and this makes nonlinear problems substantially more difficult than linear ones.

#### 5. Repeated Significance Tests for a Normal Mean

The first steps towards developing corrected diffusion approximations for first crossings of nonlinear boundaries have been taken by Hogan [9], who gives partial generalizations of (36) and (37). This section describes one possible application.

Let  $x_1, x_2, \dots$  be independent  $N(\mu, 1)$  random variables, and put  $S_n = x_1 + \dots + x_n$ . A (modified) repeated significance test of  $H_0: \mu=0$  against  $H_1: \mu \neq 0$  is defined as follows. Let  $0 < c \leq b < \infty$ ,  $0 < m_0 < m_1 < \infty$ , and define

$$(54) \quad T = \inf\{n: n \geq m_0, |S_n| \geq bn^{\frac{1}{2}}\}.$$

Stop sampling at  $\min(T, m)$  and reject  $H_0$  if either  $T \leq m$  or  $T > m$  and  $|S_m| > cm^{\frac{1}{2}}$ . The design parameters  $c, b, m_0$ , and  $m$  are customarily

chosen to give a desired significance level and desired power at some chosen  $\mu \neq 0$ , i.e. to give prescribed values of

$$(55) \quad P_{\mu}\{T \leq m\} + P_{\mu}\{T > m, |S_m| \geq cm^{\frac{1}{2}}\} \\ = P_{\mu}\{|S_m| \geq cm^{\frac{1}{2}}\} + P_{\mu}\{T \leq m, |S_m| < cm^{\frac{1}{2}}\}$$

for  $\mu=0$  and some chosen  $\mu \neq 0$ . Usually either  $c = b = b_1$ , say, and (55) simplifies to

$$P_{\mu}\{T \leq m\} ,$$

or  $b$  is slightly larger than  $b_1$  and  $c$  considerably smaller, so that the test has essentially the same power function as a fixed sample test. See [23] for a thorough discussion.

In light of (36) and (37) a natural conjecture is the following. (See also [4].) Suppose  $m_0, m \rightarrow \infty$  so that the ratio  $m/m_0$  remains constant, and assume that  $\mu$  is proportional to  $m^{-\frac{1}{2}}$ . Define

$$(56) \quad \tilde{T} = \inf\{t: t \geq m_0, |W(t)| \geq bt^{\frac{1}{2}} + \rho_+\} ,$$

where  $\rho_+$  is as defined in Section 2 and has approximately the numerical value .583. Then as  $m \rightarrow \infty$  (with  $b, c$  fixed)

$$(57) \quad P_{\mu}\{T < m, |S_m| < cm^{\frac{1}{2}}\} = P_{\mu}\{\tilde{T} < m, |W(m)| < cm^{\frac{1}{2}}\} + o(m^{-\frac{1}{2}}) .$$

Hogan's elegant argument [9] lends considerable credence to (57). It should be noted, however, that although Hogan's results are not restricted to normal populations, they do require that the  $x_i$  have vanishing third central moment. In the contrary case the form of a corrected diffusion approximation for nonlinear boundaries appears to be much more complicated.

To use (57) one must compute numerically or by means of some approximation the boundary crossing probability for Brownian motion which appears on the right hand side of (57). Techniques for obtaining such approximations have been developed in [10], [11], and [19]. For our present purposes the methods described in [23] are convenient. The following result is easily proved by the methods of Theorems 4.14 and 11.1. (Note that the asymptotic normalization used here is one of large deviations.)

(58) Theorem. Suppose that  $b, c, m_0, m \rightarrow \infty$  so that  $b/m^{1/2} = \mu_1 < b/m^{1/2} = \mu_0$ , and  $c/m^{1/2} = \gamma \leq \mu_1$  are fixed constants. Then for  $\mu > 0$

$$(59) \quad P_\mu \{ \tilde{T} < m, |W(m)| < cm^{1/2} \} \sim \exp[-m^{1/2} \mu(b-c) - \rho_+ b^2/mc] \phi(b-m^{1/2}\mu) b/cm^{1/2}\mu,$$

and

$$(60) \quad P_0 \{ \tilde{T} < m, |W(m)| < cm^{1/2} \} = 2b\phi(b) \int_{\mu_1^2 \gamma^{-1}}^{\mu_0} x^{-1} \exp(-\rho_+ x) dx \\ + 2b^{-1} \phi(b) \int_{\mu_1^2 \gamma^{-1}}^{\mu_0} x^{-1} \exp(-\rho_+ x) [\mu_1^{-2} x^2 - 1 + 3\rho_+ x - 2\rho_+^2 x^2 \\ - 2\mu_1^{-2} \rho_+ x^3 + \frac{1}{2} \mu_1^{-2} \rho_+^2 x^4] dx \\ + b^{-1} \phi(b) \exp(-\rho_+ \mu_0) (\rho_+ \mu_0 - 1) [(\mu_0/\mu_1)^2 - 3] + o[b^{-1} \phi(b)].$$

Remark. The expansion (60) is given in a convenient form for numerical integration. An alternative form is perhaps preferable if tables of the exponential integral and exponential function or incomplete gamma function are to be used.

The approximation of  $P_\mu \{ T < m, |S_m| < cm^{1/2} \}$  by the right hand side of (59) when  $\mu > 0$  is essentially the same approximation as that suggested on



the basis of a direct large deviation analysis of (55) (cf. [20] for  $c=b$  and [22] for  $c < b$ ). It is easily seen to be very accurate, at least in the case  $c=b$ , by comparing it to the exact numerical computations of Pocock [17]. The large deviation approximation of  $P_0\{T < m, |S_m| < cm^{\frac{1}{2}}\}$ , to wit (cf. [22])

$$(61) \quad P_0\{T < m, |S_m| < cm^{\frac{1}{2}}\} \sim b\phi(b) \int_{\mu_1^2 \gamma^{-1}}^{\mu_0} x^{-1} v(x) dx ,$$

where  $v(x) = 2x^{-2} \exp[-2\sum n^{-1} \phi(-\frac{1}{2} x n^{\frac{1}{2}})] = \exp(-\rho_+ x) + o(x^2)$  as  $x \rightarrow 0$  (cf. (16)), is similar to the first term on the right hand side of (60). The second order terms in (60) lead one to hope that (60) is better than (61), but on the basis of the evidence given below this does not appear to be the case, except perhaps when  $m_0$  and  $m$  are quite large.

Table 1 compares the large deviation and corrected diffusion approximations to (55) in the case  $\mu=0$ . For the large deviation case (61) is used to approximate the second term on the right hand side of (55), whereas (60) is used for the corrected diffusion case. For comparison, exact numerical computations of [6] or [17] are included when available. Otherwise the results of an 8100 repetition Monte Carlo experiment  $\pm$  one standard error provide a standard of accuracy. For the Monte Carlo experiment a variation of the importance sampling scheme described in [13] was utilized for variance reduction.

Woodroffe and Takahashi [28] have calculated a second order term for the large deviation approximation to (55) when  $b=c$ . For the last two rows of Table 1, their approximation yields .111 and .064 respectively.

TABLE 1

$$P_0\{|S_m| > cm^{\frac{1}{2}}\} + P_0\{T < m, |S_m| < cm^{\frac{1}{2}}\}$$

b	c	$m_0$	m	Corrected Diffusion	Large Deviation	Exact or Monte Carlo
2.289	2.289	1	3	.044	.049	.050
2.535	2.535	1	9	.045	.050	.050
2.986	2.20	1	5	.031	.032	.032
2.413	2.00	1	5	.064	.069	.072
2.80	2.00	3	9	.049	.050	.051 $\pm$ .001
2.24	2.24	4	24	.108	.110	.114 $\pm$ .001
2.59	2.59	20	100	.057	.056	.059 $\pm$ .001

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